Exam 3
Chapter 7

Name: Solutions

Do not write your name on any other page. Answer the following questions. *Answers without proper evidence of knowledge will not be given credit.* Make sure to make reasonable simplifications. Do not approximate answers. Give exact answers. **Only scientific calculators are allowed on this exam.**

Show your work!

1 (8 points) Use the method of undetermined coefficients to find the general form of the particular solution for the differential equation

$$y^{(4)} - 5y'' + 4y = e^x - xe^{2x}.$$

(Do not solve for the undetermined coefficients.)

$r^4 - 5r^2 + 4 = 0$

$(r^2-4)(r^2-1) = 0$

$(r-2)(r+2)(r-1)(r+1) = 0$

So $y_c = c_1 e^{-2x} + c_2 e^{2x} + c_3 e^{-x} + c_4 e^x$.

First guess:

$y_p = Ae^x + Be^{2x} + Cxe^{2x}$

Adjusted and Final form:

$y_p = Axe^x + Bxe^{2x} + Cx^2e^{2x}$.

2. (8 points) Use whatever method you desire to solve the initial value problem

$$x'' + 8x' + 15x = \cos 3t;\ \ x(0) = 2, x'(0) = -3.$$

$r^2 + 8r + 15 = 0$ $\qquad x_p = A\cos 3t + B\sin 3t$

$(r+5)(r+3) = 0$ $\qquad x_p' = -3A\sin 3t + 3B\cos 3t$

$x_c = c_1 e^{-5t} + c_2 e^{-3t}.$ $\qquad x_p'' = -9A\cos 3t - 9B\sin 3t$

$$\cos 3t = x_p'' + 8x_p' + 15x_p = (-9A\cos 3t - 9B\sin 3t) + 8(-3A\sin 3t + 3B\cos 3t)$$
$$+ 15(A\cos 3t + B\sin 3t)$$

So $(1 = 6A + 24B) \times 4$

and $(0 = 6B - 24A) +$

$4 = 102B \Rightarrow B = \frac{2}{51}$ and $A = \frac{1}{102}.$

Then $x = c_1 e^{-5t} + c_2 e^{-3t} + \frac{1}{102}(\cos 3t + 4\sin 3t)\Big|_{t=0} = c_1 + c_2 + \frac{1}{102} = 2$

$x' = -5c_1 e^{-5t} - 3c_2 e^{-3t} + \frac{3}{102}(-\sin 3t + 4\cos 3t))\Big|_{t=0} = -5c_1 - 3c_2 + \frac{12}{102} = -3$

$\vdots$

3. (8 points) Use Laplace transforms to solve the initial value problem

$$x'' + x = \sin 2t;\ x(0) = 0 = x'(0).$$

$$\mathcal{L}\{x'' + x\} = \mathcal{L}\{\sin 2t\}$$

$$s^2 X(s) + X(s) = \frac{2}{s^2+4}$$

$$X(s) = \frac{2}{(s^2+4)(s^2+1)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+1}$$

So $2 = (As+B)(s^2+1) + (Cs+D)(s^2+4)$

$$= (A+C)s^3 + (B+D)s^2 + (A+4C)s + (B+4D)$$

$0 = A+C = B+D \Rightarrow A = -C,\ B = -D$

$0 = A+4C \Rightarrow 3C = 0 \Rightarrow A = C = 0$

$2 = B+4D \Rightarrow 3D = 2 \Rightarrow D = \frac{2}{3}$ and $B = \frac{-2}{3}$.

So $X(s) = \frac{2}{3}\left(\frac{1}{s^2+1} - \frac{1}{s^2+4}\right)$

and $x(t) = \mathcal{L}^{-1}\{X(s)\} = \frac{2}{3}\left(\sin t - \frac{1}{2}\sin 2t\right)$.

4. (5 points) Find the inverse Laplace transform for the function $F(s) = \frac{5s+2}{s^2+9}$.

$$\mathcal{L}^{-1}\left\{\frac{5s+2}{s^2+9}\right\} = 5\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\}$$

$$= 5\cos 3t + \frac{2}{3}\sin 3t.$$

5. (5 points) Find the inverse Laplace transform for the function $F(s) = \frac{2s+1}{s^2+6s+11}$.

$$\frac{2s+1}{s^2+6s+13} = \frac{2s+1}{(s+3)^2+4} = \frac{2(s+3)-5}{(s+3)^2+4}$$

So $f(t) = \mathcal{L}^{-1}\{F(s)\}$

$$= e^{-3t}\left(2\cos 2t - \frac{5}{2}\sin 2t\right)$$

6. (8 points) Use the fact that $\mathcal{L}\{tf(t)\} = -\frac{d}{ds}(F(s))$ to solve for $X(s)$ in the differential equation

$$tx'' + (3t-1)x' + 3x = 0; \quad x(0) = 0.$$

$$-\frac{d}{ds}\left(s^2X - x'(0)\right) - \left(3\frac{d}{ds}+1\right)(sX) + 3X = 0$$

$$-\left(2sX + s^2X'\right) - \left(3(sX'+X) + sX\right) + 3X = 0$$

$$-X(3s) \quad -X'(s^2+3s) = 0$$

$$\frac{X'}{X} = -\frac{3s}{s^2+3s} = -\frac{3}{s+3}$$

So $\displaystyle\int \frac{dX}{X} = \int -\frac{3}{s+3}\,ds$

$$\ln X = -3\ln(s+3) + C$$

$$X = C(s+3)^{-3} = \frac{C}{(s+3)^3}$$

7. (8 points) Consider an RLC circuit with $R = 100$ ohms, $L = 0$ henries, $C = 10^{-3}$ farads and $e(t) = 100t$ if $0 \le t < 1$ and $e(t) = 0$ if $t \ge 1$ volts at time t. Use the facts that $\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}F(s)$ and $\mathcal{L}^{-1}\{e^{-as}F(s)\} = u(t-a)f(t-a)$, where $\mathcal{L}\{f(t)\} = F(s)$, to solve the resulting differential equation:

$$100I'' + 1000I = e'(t)$$

for the current $I(t)$ (in amperes).

$$i'' + 10i = \begin{cases} 1, & 0 \le t < 1 \\ 0, & t \ge 1 \end{cases} = f(t).$$

$$f(t) = (1 - u(t-1))f(t)$$
$$= (1 - u(t-1))f(t-1)$$

So $\mathcal{L}\{i'' + 10i\} = \mathcal{L}\{f(t)\}$

becomes $s^2 I + 10I = \dfrac{1-e^{-s}}{s}$

and $I = \dfrac{1-e^{-s}}{s(s^2+10)}$

$$\frac{1}{s(s^2+10)} = \frac{A}{s} + \frac{Bs+D}{s^2+10} \quad \text{So} \quad 1 = A(s^2+10) + s(Bs+D)$$

$$\left.\begin{aligned} 0 &= A + B \\ 0 &= D \\ 1 &= 10A \end{aligned}\right\} \Rightarrow \begin{aligned} A &= \tfrac{1}{10} \\ B &= -\tfrac{1}{10} \end{aligned}$$

So $I = \dfrac{(1-e^{-s})}{10}\left(\dfrac{1}{s} - \dfrac{s}{s^2+10}\right)$

and $i(t) = \frac{1}{10}\left[(1 - \cos\sqrt{10}\,t) - u(t-1)(1 - \cos\sqrt{10}(t-1))\right]$